Density of states for almost-diagonal random matrices

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We study the density of states (DOS) for disordered systems whose spectral statistics can be described by a Gaussian ensemble of almost-diagonal Hermitian random matrices. The matrices have independent random entries $H_{i\geqslant j}$ with small off-diagonal elements: $\langle|H_{i\neq j}|^2\rangle \ll \langle|H_{ii}|^2\rangle \sim 1$. Using the recently suggested method of a virial expansion in the number of interacting energy levels [J. Phys. A **36**, 8265 (2003)], we calculate the leading correction to the Poissonian DOS in the cases of the Gaussian orthogonal and unitary ensembles. We apply the general formula to the critical power-law banded random matrices and the unitary Moshe-Neuberger-Shapiro model and compare the DOS's of these models.

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Recently, extensive attention has been devoted to unconventional random matrix theories (RMT's) that interpolate between the Wigner-Dyson RMT and banded RM (BRM) with the (almost) Poissonian level statistics and can be used as a helpful tool to explore the localization transition. One of these models is the power law banded random matrix (PLBRM) theory [1–3] for which the variance of the off-diagonal elements reads

$$\text{PLBRM:}\quad \langle|V_{ij}|^2\rangle = \frac{1}{2}\,\frac{1}{1+\left[\frac{N}{\pi}\sin\left(\frac{\pi}{N}|i-j|\right)\right]^{2\alpha}\Big/\, b^{2\alpha}}. \tag{1}$$

It is nearly constant inside the band $|i-j|<\lambda\sim b$, and decreases as a power-law function $\langle|V_{ij}|^2\rangle\sim 1/|i-j|^{-2\alpha}$ for $|i-j|>\lambda$. Equation (1) is written for periodic boundary conditions of the PLBRM Hamiltonian. The special case $\alpha=1$ is relevant for description of critical systems with multifractal eigenstates [1–5], in particular for systems at the Anderson localization-delocalization transition point. On the other hand, it has been conjectured [6] that the spectral statistics of critical PLBRM with large b can be mapped onto the Calogero-Sutherland (CS) model [7] at low temperature where instead of the spectral problem one studies the statistics of interacting (for the real off-diagonal elements in PLBR) or noninteracting (for the complex off-diagonal elements in PLBR) fermions in a parabolic confinement potential. The case $\alpha>1$ corresponds to the power-law localization which can be found in certain periodically driven quantum-mechanical systems [8]. If $\alpha\leqslant 1/2$ the spectral statistics of PLBRM approaches the Wigner-Dyson universality class with $\beta=1$ or 2.

The exactly solvable model of Moshe, Neuberger, and Shapiro (MNS) also incorporates both the Poissonian and the Wigner-Dyson level statistics [9]. The probability distribution of the Hamiltonian $\hat{\mathcal{H}}$ in MNS is given by $P(\hat{\mathcal{H}}) = \int d\hat{U}\ \mathcal{P}_{\hat{U}}(\hat{\mathcal{H}})$, where

$$\mathcal{P}_{\hat{U}}(\hat{\mathcal{H}}) \propto \exp\left(-\mathrm{Tr}\hat{\mathcal{H}}^2 - \left(\frac{N}{2\pi b}\right)^2 \mathrm{Tr}([\hat{U},\hat{\mathcal{H}}][\hat{U},\hat{\mathcal{H}}]^\dagger)\right); \tag{2}$$

the matrix $\hat{U}$ is either unitary for complex Hermitian matrices $\hat{\mathcal{H}}$ (the unitary MNS) or orthogonal for real symmetric matrices $\hat{\mathcal{H}}$ (orthogonal MNS) and $d\hat{U}$ is the Haar measure.

The spectral properties of the unitary MNS turn out to be equivalent to a system of noninteracting one-imensional ($1D$) fermions in a parabolic confinement [9]. The spectral statistics of the orthogonal MNS coincides [10] with the statistics of $1D$ fermions in a parabolic potential with the long-range attractive interaction $\propto(x_i-x_j)^{-2}$. This model of strongly correlated fermions is a particular case of the Calogero-Sutherland model which has been intensively studied as a toy model for the fractional statistics. In both cases the parameter b of MNS corresponds to the inverse temperature of CS: $b\sim 1/T_{\mathrm{CS}}$.

The connection between the two models is especially clear in the unitary case where the unitary matrix $\hat{U} = M\ \mathrm{diag}\{e^{i\varphi_i}\}\ M^\dagger$ can be diagonalized by a unitary transformation. Then the variances of $V_{i,j}=(M^\dagger\hat{\mathcal{H}}M)_{i,j}$ in MNS are given by

$$\text{MNS:}\quad \langle|V_{ij}|^2\rangle = \frac{1}{2}\,\frac{1}{1+\left(\frac{N}{\pi b}\right)^2 \sin^2\left(\frac{\varphi_i-\varphi_j}{2}\right)}. \tag{3}$$

One can easily see that Eq. (3) coincides with Eq. (1) at $\alpha=1$ if the phases $\varphi_n = 2\pi n/N$ are arranged as an ordered array on a circle. In general the MNS model can be considered as

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an extension of the PLBRM model for the case of the arbitrary arrangement of phases φ_n randomly distributed over the circle.

The following formula is valid to calculate the averaged value of an observable $A(\hat{H})$ which is invariant under the transformation $\hat{H}\to M^{\dagger}\hat{\mathcal{H}}M$:

$$\langle\langle A\rangle_{\hat{H}}\rangle_{\hat{U}}\equiv\frac{\int\langle A\rangle_{\hat{H}}\,P(\{\varphi_i\})D\{\varphi_i\}}{\int P(\{\varphi_i\})D\{\varphi_i\}}. \tag{4}$$

Here $P(\{\varphi\})$ is the joint probability distribution of phases [9]

$$P(\{\varphi\})\sim\prod_{i>j}\frac{\sin^2\left(\frac{\varphi_i-\varphi_j}{2}\right)}{1+\left(\frac{N}{\pi b}\right)^2\sin^2\left(\frac{\varphi_i-\varphi_j}{2}\right)} \tag{5}$$

and $\langle A\rangle_{\hat{H}}$ stands for the averaging over the Gaussian random matrix $\hat{H}$ with entries having zero mean value and the variance given by Eq. (3).

The two-point correlation function, which follows from Eq. (5) after the integration over all but two phases, was calculated by Gaudin with the help of the model of free noninteracting fermions with a linear spectrum [11]:

$$\mathcal{R}_2(s)=1-\frac{1}{(2\pi b)^2}\left|\int_{-\ln(e^{2\pi b}-1)}^{\infty}\frac{e^{\mathbf{i}(\omega s/b)}\,d\omega}{e^{\omega}+1}\right|^2,$$

$$s=\phi_i-\phi_j\equiv(\varphi_i-\varphi_j)(N/2\pi). \tag{6}$$

If $|s|\gg b$, the correlation function is almost constant $\mathcal{R}_2(|s|\gg b)\to 1$. There is a repulsion between phases at a small scale controlled by b: $\mathcal{R}_2(|s|\ll b)\sim(s/b)^2$.

For $b\gg 1$ the spectral statistics of the critical PLBRM with $\alpha=1$ and MNS are asymptotically the same and at $b\to\infty$ they approach the Wigner-Dyson statistics [2,3,6]. This is because the phase repulsion in MNS is strong at large b. The phases $\phi_{i,j}$ form an approximately equidistant latticelike structure [2]. In the opposite case $b\ll 1$, the phase repulsion in MNS is weak and the phases $\phi_{i,j}$ do not form a regular structure. Disorder in the phase arrangements at small distances $|\phi_i-\phi_j|\sim 1$ may become especially important. Therefore, there is no evident correspondence between critical PLBRM and MNS at $b\ll 1$ even though both ensembles have the same first correction to the Poissonian level rigidity [3,12] and the numerics have revealed a relatively small difference in the level rigidity of PLBRM and MNS at $b\sim 1$ [12].

The progress in BRM and PLBRM theories became possible because of mapping [1,13] onto the nonlinear supersymmetric σ-model [14] that allowed one to obtain rigorous results by using various powerful methods of the field theory. However, the σ-model always starts from delocalized (i.e., diffusive) modes and such mapping is only justified if the bandwidth $\lambda\gg 1$. In the opposite case where all the off-diagonal matrix elements are parametrically small compared to the diagonal ones and the system is close to the localization, no field-theoretical approach is known so far. Yet such *almost-diagonal* RMT's may possess nontrivial properties because of the slow decay of the off-diagonal matrix elements $\langle|H_{ij}|^2\rangle$ with increasing $|i-j|$. For instance, it is of fundamental interest to study the spectral statistics in systems with power-law localization that takes place in the power-law banded random matrix ensembles at $\alpha>1$. Another problem to study is the critical almost-diagonal PLBRM. It is known that the eigenvectors of PLBRM with $\alpha=1$ remain multifractal for an arbitrary small value of λ [5]. This means that the typical eigenfunction is extended though very sparse at small λ. Their fractal dimensions are small as compared to the dimension $d=1$ of the underlying chain with the long-range hopping. Thus almost-diagonal PLBRM's may display a localization-delocalization transition with changing the exponent α as well as their large bandwidth counterpart [1].

Recently, we have suggested a method that allows us to study spectral statistics of a disordered system described by an ensemble of the almost-diagonal random matrices [15]. It is a *virial expansion* in the number of interacting energy levels. Unlike the field-theoretical approach, the virial expansion starts from the Poissonian statistics and yields a regular expansion in powers of the small parameter controlling the ratio of the off-diagonal elements to the diagonal ones $\langle|H_{i\neq j}|^2\rangle/\langle H_{ii}^2\rangle\sim b^2\ll 1$. The expansion has been represented by the summation of diagrams which are generated with the help of the Trotter formula. A rigorous selection rule has been established for the diagrams, which allows us to account for exact contributions of a given number of resonant and nonresonant interacting levels. The method offers a controllable way to find an answer to the question of when a weak interaction of levels can drive the system from localization toward criticality and delocalization. An example of the spectral form factor has been considered for a generic dependence of the variance $\langle|H_{i\neq j}|^2\rangle$ on the difference $i-j$. It has been shown that a term of the order of b^{c-1} is governed by the interaction of c energy levels. The general theory has been applied to the Rosenzweig-Porter [16] model and to the critical PLBRM.

In the present paper, we continue studying the spectral statistics with the help of the virial expansion. We calculate the density of states (DOS) for the ensembles of the Gaussian almost-diagonal random matrices. Based on the detailed presentation of the method in Ref. [15], we will explain the corresponding diagrammatic technique for DOS. We derive the leading correction to the Poissonian DOS for the models of critical PLBRM and MNS.

Let us consider a Hermitian random matrix (RM) of size $N\times N$, $N\gg 1$, from a Gaussian ensemble. We assume that the matrix entries are random and independent. The RM is the Hamiltonian $\hat{H}$ of the matrix Schrödinger equation $\hat{H}\psi_n=\epsilon_n\psi_n$, where ϵ_n and ψ_n are the eigenvalues and eigenvectors, respectively. We define statistical properties of the matrix entries as

$$\langle H_{i,j}\rangle = 0,\ \langle H_{i,i}^2\rangle = \frac{1}{\beta},\ \langle |H_{i,j}|^2\rangle = b^2\ \ \mathcal{F}(|i-j|),\quad i\neq j, \tag{7}$$

where $\mathcal{F}(|i-j|)>0$ is a smooth function of its argument, and the parameter b is small:

$$b \ll 1.$$

The condition $b\ll 1$ means that RM is almost diagonal. The parameter β corresponds to the Dyson symmetry classes $\beta_{\mathrm{GOE}}=1$, $\beta_{\mathrm{GUE}}=2$. The angular brackets denote the ensemble averaging.

We will study the spectral properties of the system concentrating on the ensemble averaged density of states

$$\rho(E) = \left\langle \sum_n\ \delta(E-\epsilon_n)\right\rangle. \tag{8}$$

For almost-diagonal RM the representation of spectral statistics in the time domain is more convenient [15] therefore we explore below the Fourier transformed DOS as a function of time:

$$C(t) = \langle \mathrm{Tr} e^{\mathbf{i}\hat{H}t}\rangle. \tag{9}$$

We start with a brief explanation of the method of the virial expansion that has been developed in detail in Ref. [15]. As far as we investigate the properties of almost-diagonal RM's the Hamiltonian can be naturally divided into a diagonal part $\hat{H}_\varepsilon$ and a matrix of hopping elements $\hat{V}$:

$$\hat{H} \equiv \hat{H}_\varepsilon + \hat{V}. \tag{10}$$

For a strictly diagonal matrix ($\hat{V}=0$) the Poissonian DOS $C_P(t)$ is calculated straightforwardly from Eq. (9):

$$C_P(t) = Ne^{-t^2/2\beta}. \tag{11}$$

It follows from the definition (7) that the hopping elements $H_{i,j}\equiv V_{i,j}\sim b$ are small compared to the diagonal ones $H_{i,i}\equiv\varepsilon_i\sim 1$. However, a direct expansion of the exponentials $e^{\mathbf{i}(\hat{H}_\varepsilon+\hat{V})t}$ in Eq. (9) in terms of $\hat{V}$ would involve serious difficulties because the matrices $\hat{H}_\varepsilon$ and $\hat{V}$ do not commute with each other. One possible way to overcome these problems is to represent $e^{\mathbf{i}(\hat{H}_\varepsilon+\hat{V})t}$ as a product of exponentials containing matrices $\hat{H}_\varepsilon$ and $\hat{V}$ separately. We do this using the Trotter formula [17]

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty}(e^{\hat{A}/n}\ e^{\hat{B}/n})^n \Rightarrow e^{\mathbf{i}\hat{H}t} = \lim_{n\to\infty}\prod_{p=1}^{n}(e^{\mathbf{i}\hat{H}_\varepsilon t/n}e^{\mathbf{i}\hat{V}t/n}). \tag{12}$$

In order to obtain corrections to $C(t)$ proportional to $(bt)^m$ one has to expand m different exponentials in the infinite product in the right-hand side (RHS) of Eq. (12), $e^{\mathbf{i}t\hat{V}/n}\simeq 1+\mathbf{i}(t\hat{V}/n)$, setting in the rest $n-m$ exponentials $\hat{V}\to 0$, and to perform the Gaussian averaging over $\hat{H}_\varepsilon$ and $\hat{V}$. The terms with an odd power m give zero after the disorder averaging. Thus, the power m must be even and we can substitute $m\to 2k$.

There are $n!/2k!(2k-n)!$ ways to choose $2k$ exponentials to be expanded in $\hat{V}$ from the RHS of Eq. (12). Therefore, before taking the limit $n\to\infty$, one has to account for all possible different arrangements of the expanded exponentials which results in a summation over the Trotter variables $p_1,p_2,\ldots,p_{2k-1}$; $\Sigma_{l=1}^{2k-1}p_l\equiv n$. Each discrete variable p_l denotes the number of successive exponentials $e^{\mathbf{i}\hat{H}_\varepsilon t/n}$ fused together:

$$\left\langle \cdots e^{\mathbf{i}\hat{H}_\varepsilon t/n}\left(1+\mathbf{i}\frac{t\hat{V}}{n}\right)\underbrace{\times e^{\mathbf{i}\hat{H}_\varepsilon t/n}\cdots e^{\mathbf{i}\hat{H}_\varepsilon t/n}\times}_{p_l\ \text{exponentials where}\ \hat{V}\to 0}\times\left(1+\mathbf{i}\frac{t\hat{V}}{n}\right)e^{\ \mathbf{i}\hat{H}_\varepsilon t/n}\cdots\right\rangle$$
$$=\left\langle \cdots e^{\mathbf{i}\hat{H}_\varepsilon t p_{l-1}/n}\left(1+\mathbf{i}\frac{t\hat{V}}{n}\right)e^{\ \mathbf{i}\hat{H}_\varepsilon t p_l/n}\left(1+\mathbf{i}\frac{t\hat{V}}{n}\right)\times e^{\mathbf{i}\hat{H}_\varepsilon p_{l+1}t/n}\cdots\right\rangle.$$

We can introduce scaled variables $Y_l=p_l/n$ converting the summation over p_l to the integration over Y_l and eliminating the parameter n from further calculations. The resulting expression must be averaged over the diagonal elements ε_i and yields the integral $\mathcal{I}_\beta(t,k)$ which depends on time t, power k, and parameter β.

For almost diagonal RM's, the higher the number of interacting energy levels the smaller the correction to the Poissonian level statistics governed by that interaction. In particular, to find *the leading in b correction* $C_1(t)$ to $C_P(t)$,

$$C(t)\simeq C_P(t)+C_1(t)+\cdots,\ \ C_1(t,\ b\ll 1)\ll C_P(t),$$

we retain in the obtained series all terms that correspond to an interaction of two different species of the diagonal elements ε_i and ε_j via the hopping elements $V_{i,j}$ and $V_{j,i}$ for any indices $i\neq j$ in the range from 1 to N. Then at fixed k we find the following contribution to the correction $C_1(t)$:

$$C_1^{(k)} = (\mathbf{i}\ t)^{2k}\ \mathcal{I}_\beta(t,k)\ \sum_{i\neq j}^{N}\ \langle (V_{i,j}V_{j,i})^k\rangle.$$

After integration over $2k-2$ Trotter variables $\{Y_1,Y_2,\ldots Y_{2k-2}\}$, $\mathcal{I}_\beta$ can be simplified to one-dimensional integral

$$\mathcal{I}_\beta(t,k) = \frac{e^{-t^2/4\beta}}{k!\ \ (k-1)!}\int_0^{1/2}(1/4-Y^2)^{k-1}e^{-(tY)^2/\beta}dY. \tag{13}$$

In accordance with the definition (7), the Gaussian average of $\langle (V_{i,j}V_{j,i})^k\rangle$ can be transformed to the following form:

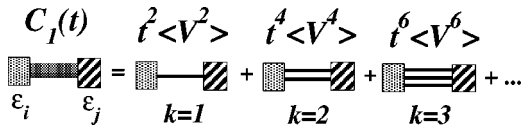

FIG. 1. Graphic illustration of the series (16) for the leading correction $C_1(t)$ to the Poissonian DOS $C_P(t)$. Boxes with different patterns mark the energy levels ε_i and ε_j. In each diagram with a given k, they are connected by the k interaction lines which are associated with the factor $t^{2k}\langle \hat{V}^{2k}\rangle \sim (tb)^{2k}$.

$$\langle (V_{i,j}V_{j,i})^k\rangle = \mathcal{K}_\beta(k)\times b^{2k}\ \ \mathcal{F}^k(|i-j|), \tag{14}$$

where $\mathcal{K}_\beta(k)$ is the combinatorial factor. Due to the Wick theorem, it is equal to factorials

$$\mathcal{K}_\beta(k)=\begin{cases}(2k-1)!!, & \beta=1,\\ k!, & \beta=2.\end{cases} \tag{15}$$

We have to sum the contributions $C_1^{(k)}$ over the parameter k at the end, $C_1=\Sigma_k C_1^{(k)}$. This summation yields the answer for $C_1(t)$ as a series in powers of the product (bt):

$$C_1(t)=N\sum_{k=1}^{\infty}\ (-1)^k(bt)^{2k}\ \mathcal{I}_\beta(t,k)\ \mathcal{K}_\beta(k)\ \mathcal{R}_N(k),$$

$$\mathcal{R}_N(k)\equiv\frac{1}{N}\sum_{i\neq j}^{N}\ \mathcal{F}^k(|i-j|) \tag{16}$$

(see the graphic presentation in Fig. 1). We will show below that C_1 is not larger than $O(b^1)$. We emphasize that neither the combinatorial factor $\mathcal{K}_\beta(k)$ nor the integral over the Trotter variables $\mathcal{I}_\beta(t,k)$ depend on the correlation function $\mathcal{F}(|i-j|)$. Thus, they are universal. The factor $\mathcal{R}_N(k)$ is, on the contrary, model dependent. It arises because of summation of the product of the correlation functions $\mathcal{F}(|i-j|)$ over the indices i and j and is not universal. If we associate the Hamiltonian $\hat{H}$ with a one-dimensional chain having long-range hopping between different sites, the summation over i and j turns out to be the summation in the real space along the sites of the chain. As the function $\mathcal{F}(|i-j|)$ depends only on the difference of $|i-j|$ the leading part of the real space sum is

$$R_N(k)\simeq 2\sum_{m=1}^{\infty}\ \mathcal{F}^k(m)+O(1/N). \tag{17}$$

In what follows we will neglect the correction of the order $O(1/N)$ to $R_N(k)$.

To further simplify the analysis of DOS we insert Eqs. (13), (15), (17) into the series (16) and change the summation order. At first, we sum over the power k getting the answer for $C_1(t)$ as a one-dimensional series over m, i.e., as the sum in the real space

$$C_1(t)=-2Ne^{-\frac{t^2}{4\beta}}\sum_{m=1}^{\infty}\ Z(bt,m)\ \ \tilde{\mathcal{I}}_\beta[t,Z(bt,m)],$$

$$Z(bt,m)\equiv(bt)^2\mathcal{F}(m), \tag{18}$$

$$\tilde{\mathcal{I}}_\beta(t,Z)\equiv\int_0^{1/2}dY\ \ e^{-[(tY)^2/\beta+(1/4-Y^2)Z]}$$

$$\times\begin{cases} I_0\left[\left(\frac{1}{4}-Y^2\right)Z\right]-I_1\left[\left(\frac{1}{4}-Y^2\right)Z\right], & \beta=1,\\ 1, & \beta=2.\end{cases} \tag{19}$$

An analytical integration over Y is easily doable at $\beta=2$ and the integral (19) simplifies to the error function

$$\tilde{\mathcal{I}}_{\beta=2}(t,Z)=e^{-Z/4}\sqrt{\frac{\pi}{2(t^2-2Z)}}\ \ \mathrm{erf}\left(\sqrt{\frac{t^2-2Z}{8}}\right). \tag{20}$$

Let us consider short- and long-time asymptotics. If $bt\ll 1$, C_1 is determined by a diagram with the minimal number of the interaction lines (see Fig. 1), i.e., we can keep in the power series (16) the single term with $k=1$ having $\mathcal{K}_\beta(1)=1$ and $\mathcal{I}_\beta(t,1)=(\sqrt{\pi\beta}/2|t|)\exp(-t^2/4\beta)\times\mathrm{erf}(t/\sqrt{4\beta})$:

$$C_1(bt\ll 1)\simeq -Nb^2\frac{\sqrt{\pi\beta}}{2}\,|t|\exp\left(-\frac{t^2}{4\beta}\right)\ \mathrm{erf}\left(\frac{t}{\sqrt{4\beta}}\right)R_N(1), \tag{21}$$

which at $t<\sqrt{4\beta}$ is parametrically smaller than C_P: $C_1/C_P|_{bt\ll 1}\sim b\times(bt)$.

One can do the Fourier transform of Eq. (21) and show that for large energies $\varepsilon\gg 1$ the correction to the tail of the DOS has the same Gaussian exponential dependence as the distribution of diagonal matrix elements of $\hat{H}$ unless $R_N(1)$ is divergent:

$$\rho(\varepsilon\gg 1)\approx N\sqrt{\frac{\beta}{2\pi}}\ \ e^{-(\beta/2)\varepsilon^2}\ \ [1+b^2\ \ \beta\ \ R_N(1)]. \tag{22}$$

Thus we conclude that there is no slowly decaying Lifshitz tails for almost diagonal PLBRM with $\alpha>1/2$ (including the critical PLBRM with $\alpha=1$) even though the multipoint correlation functions may significantly deviate from the Poisson distribution.

If $t\gg 1$, the integral $\mathcal{I}_\beta$ can be calculated approximately by substituting the Dirac δ function for the exponential in the integrand of Eq. (13):

$$e^{-(tY)^2/\beta}\to\frac{\sqrt{\beta\pi}}{|t|}\ \ \delta(Y)\Rightarrow\mathcal{I}_\beta(t,k)\simeq e^{-t^2/4\beta}\ \left(|t|^{-1}\frac{\sqrt{\beta\pi}}{2^{2k-1}k!\ \ (k-1)!}+O(1/|t|^2)\right), \tag{23}$$

and we obtain a simplified version of the series (18):

$$C_1(t\gg 1)\simeq -N\ \sqrt{\beta\pi}\ e^{-t^2/4\beta}\sum_{m=1}^{\infty}\frac{Z(bt,m)}{|t|}\ e^{-Z(bt,m)/4}\times\begin{cases} I_0[Z(bt,m)]-I_1[Z(bt,m)], & \beta=1,\\ 1, & \beta=2.\end{cases} \tag{24}$$

We note that $Z(bt,m)/|t|\sim b\times(bt)$ and, thus, we can schematically rewrite the asymptotic expression (24) as $C_1(t\gg 1)\simeq b\times\tilde{C}_1(bt)$. If the function $\tilde{C}_1(bt)$ is finite at any value of (bt) the correction C_1 is again parametrically smaller than C_P (see examples below).

Now we will apply the general formulas (16), (18) for the specific models of our interest with the definite correlation function $\mathcal{F}(|i-j|)$.

The model of Power law banded random matrices. The model of PLBRM is defined by Eq. (1). We restrict ourselves to the critical and almost-diagonal PLBRM with $\alpha=1$ and $b\ll 1$ so that the variance (1) simplifies to the following form [18]:

$$\langle|V_{ij}|^2\rangle\simeq\frac{1}{2}\frac{b^2}{\left(\frac{N}{\pi}\right)^2\sin^2\left(\frac{\pi}{N}|i-j|\right)}+O(b^4). \tag{25}$$

The term of $O(b^4)$ is not important for the correction C_1. We neglect this term below.

We define the correlation function $\mathcal{F}$ in Eq. (7) for a critical almost-diagonal PLBRM:

$$\mathcal{F}(|i-j|)=\frac{1}{2}\ \frac{1}{\left(\frac{N}{\pi}\right)^2\sin^2\left(\frac{\pi}{N}|i-j|\right)},\quad i\neq j. \tag{26}$$

Next we note that the sum in the real space R_N, Eq. (17), is governed by small distances $m\ll N$ and, therefore, its leading term does not depend on the boundary conditions for the underlying chain with the long-range hopping

$$\sum_{m=1}^{N}\mathcal{F}^k(m)=\sum_{m=1}^{N}\frac{1}{2^k}\frac{1}{\left(\frac{N}{\pi}\right)^{2k}\sin^{2k}\left(\frac{\pi}{N}m\right)}\simeq\sum_{m=1}^{\infty}\frac{1}{(2\ m^2)^k}+O(1/N). \tag{27}$$

Substituting Eq. (27) into the series (18) we find the correction C_1:

$$C_1(t)=-Ne^{-t^2/4\beta}\sum_{m=1}^{\infty}\left(\frac{bt}{m}\right)^2\tilde{\mathcal{I}}_\beta\left[t,\frac{1}{2}\left(\frac{bt}{m}\right)^2\right]. \tag{28}$$

Let us consider the limiting cases of small and large time. If $bt\ll 1$, we insert $\mathcal{R}_N(1)=\pi^2/6$ into Eq. (21) and arrive at

$$C_1(bt\ll 1)\simeq -N\frac{\pi^{5/2}\sqrt{\beta}}{12}b^2|t|\ \exp\left(-\frac{t^2}{4\beta}\right)\ \mathrm{erf}\left(\frac{t}{\sqrt{4\beta}}\right). \tag{29}$$

We emphasize that the sum in real space $\mathcal{R}_N(1)=\Sigma_m m^{-2}$ converges at $m\sim 1$. Thus, when time t is small compared to b^{-1} the correction to the Poissonian DOS of the critical PLBRM is sensitive to the behavior of the correlation function $\mathcal{F}(m)$ at short distances. This statement is, in fact, more general and holds true for any ensemble of the almost-diagonal PLBRM, where $\alpha>1/2$ and $\mathcal{R}_N(1)$ converges in the thermodynamical limit $N\to\infty$ [19].

If $t\gg 1$, we substitute the correlation function (26) into the series (24) which converges at $m\sim tb$. If the product (tb) is large the correction C_1 is not sensitive to the short distances $m\sim 1$ and we can replace the sum over m by the integral and find the asymptotics at the long time $tb\gg 1$:

$$C_1(tb\gg 1)\simeq -Nbe^{-t^2/4\beta}\times\begin{cases}2, & \beta=1,\\ \pi, & \beta=2.\end{cases} \tag{30}$$

The Moshe-Neuberger-Shapiro model. The model of MNS is defined by Eq. (3). Let us consider the unitary case with $\beta=2$. From Eqs. (3) and (7) we find the function $\mathcal{F}$ for MNS:

$$\mathcal{F}(|\phi_i-\phi_j|)=\frac{1}{2}\frac{1}{b^2+\left[\frac{N}{\pi}\sin\left(\frac{\pi}{N}|\phi_i-\phi_j|\right)\right]^2}. \tag{31}$$

If $b\ll 1$, the Gaudin correlation function of the phases $\phi_{i,j}$, Eq. (6), simplifies to the following form:

$$\mathcal{R}_2(s/b)\simeq\frac{(s/b)^2}{1+(s/b)^2}. \tag{32}$$

The condition $\langle|V_{i,j}|^2\rangle\ll 1$ holds true in MNS if $|\phi_i-\phi_j|\gg 1$. However, in contrast to the PLBRM case where the minimal distance $|\phi_i-\phi_j|=1$, it is violated inside the band $0\leq|\phi_i-\phi_j|\leq b$ where the matrix $\hat{H}$ is no longer almost diagonal. This band is, however, narrow at $b\ll 1$. Therefore, the contribution of this band to the DOS averaged over phases ϕ_i is small in the parameter b.

We apply the strategy of the virial expansion to calculate the average over $\hat{H}$ at the fixed phases ϕ_i, see Eq. (4). The phase averaging is done at the last step and it reduces the sum in real space to an integral over the difference of two phases:

$$\langle R_N(k)\rangle_{\phi_i}\simeq 2\int_0^\infty\mathcal{F}^k(s)\ \ \mathcal{R}_2(s)\ \ \mathrm{d}s+O(1/N). \tag{33}$$

The case where the correlation function $\mathcal{F}$ depends on the difference of the integer indices can be restored from Eq. (33) by substituting a sum of the δ functions instead of the two-point correlator: $\mathcal{R}_2(s)\to\Sigma_{m=1}^{\infty}\delta(s-m)$. In full analogy with PLBRM we can prove that the leading term of the function $\langle R_N(k)\rangle_{\phi_i}$ in MNS does not depend on the boundary conditions and transform Eqs. (31)–(33) to a simpler form

$$\langle R_N(k)\rangle_{\phi_i} \simeq 2\int_0^\infty \left(\frac{1}{2}\frac{1}{b^2+s^2}\right)^k \mathcal{R}_2(s)ds. \qquad (34)$$

Substituting Eq. (32) at $b \ll 1$ into formula (34) we find

$$\langle R_N(k)\rangle_{\phi_i} \simeq \frac{2b}{(2b^2)^k}\int_0^\infty \frac{S^2}{(1+S^2)^{k+1}}\, dS. \qquad (35)$$

We insert Eq. (35) into series (16) and derive an analog of Eq. (18) for MNS

$$C_1(t) = -Nbe^{-t^2/8}\int_0^\infty \frac{(tS)^2}{(1+S^2)^2}\, \tilde{\mathcal{I}}_{\beta=2}\left(t, \frac{t^2}{1+S^2}\right) dS, \qquad (36)$$

which reduces to the following expression:

$$C_1(t) = -\pi Nbe^{-t^2/8}(1-e^{-t^2/8}). \qquad (37)$$

The answer (37) coincides with the leading in b term of the Fourier transformed DOS of MNS obtained from the model of noninteracting $1D$ fermions in a parabolic confinement [9].

Let us discuss the results obtained by means of the virial expansion for PLBRM in GUE [Eq. (28) with $\beta=2$] and MNS [Eq. (36)]. First we note that at large time scales, $tb \gg 1$, the summation over the real space converges at large distances ($m \sim tb$ in PLBRM and $s = Sb \sim tb$ in MNS). In this case, we can simplify the integrand in the right-hand side of Eq. (34) by ignoring b in the denominator and putting $\mathcal{R}_2 \to 1$. We immediately see that the leading term resulting from the sum over m in Eq. (24) is equal to the one originating from the integral over S in Eq. (36). Thus, we conclude that the first corrections to the Poissonian DOS coincide for PLBRM and MNS in the long-time limit [20]:

$$tb \gg 1 \Rightarrow \frac{C_1(\beta=2)|_{\mathrm{PLBRM}}}{C_1|_{\mathrm{MNS}}} \simeq 1.$$

The situation is quite different in the opposite limit of the short-time scale $tb \ll 1$, where the asymptotics for C_1 is governed by the single diagram with $k=1$. This diagram is highly sensitive to the behavior of the function $\mathcal{F}$ at short distances and, therefore, yields absolutely different answers for PLBRM and MNS after the summation in real space. At a fixed value of m or s the leading diagram is of the order of $\sim b^2$. In the case of PLBRM the sum over m reduces to a numerical prefactor in Eq. (29) leaving the power of b^2 unchanged. The integration over s in the case of MNS is strongly affected by the region $0 \leqslant s \leqslant b$ where $b^2\mathcal{F}(s)\,\mathcal{R}_2(s)|_{s\sim b} \sim 1$ and the off-diagonal elements of $\hat{H}$ are of the order of diagonal ones. This region makes the main contribution to the correction to DOS which is small only because of the small volume of this region $\Delta s \propto b$. Thus we find that the integration over small $s \leqslant b$ in MNS leads to a reduction of the power of b in the short-time limit compared to the PLBRM case

$$tb \ll 1 \Rightarrow \frac{C_1(\beta=2)|_{\mathrm{PLBRM}}}{C_1|_{\mathrm{MNS}}} \sim b.$$

This is a clear manifestation of the *nonequivalence* of PLBRM and MNS models for small b, or the nonequivalence of PLBRM at small b and the CS model at high temperature.

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[18] The condition $\alpha=1$ is not necessary for the application of the virial expansion to DOS and it will be removed in the forthcoming paper.
[19] If the sum in the real space diverges the virial expansion fails and we cannot consider a finite number of interacting energy levels to derive DOS.
[20] This is similar to the first correction to the Poissonian level compressibility calculated in Ref. [15]. This correction is governed by behavior of the correlation function $\mathcal{F}$ at the large distances and, therefore, is the same in PLRBM and MNS.